Suggested Solution to Assignment 4

1. (a) Consider $A\vec{y}^{T} = \begin{pmatrix} 1 & 5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 5-4y \\ 3+y \end{pmatrix}$

Now
$$\begin{cases} 5-4y > 3+y & \text{if } 0 \le y < \frac{2}{5} \\ 5-4y = 3+y & \text{if } y = \frac{2}{5} \\ 5-4y < 3+y & \text{if } \frac{2}{5} < y \le 1 \end{cases}$$

We see that

$$P = \left\{ (x,y) : \left(x=0 \wedge \tfrac{2}{5} < y \le 1 \right) \vee \left(0 \le x \le 1 \wedge y = \tfrac{2}{5} \right) \vee \left(x = 1 \wedge 0 \le y < \tfrac{2}{5} \right) \right\}$$

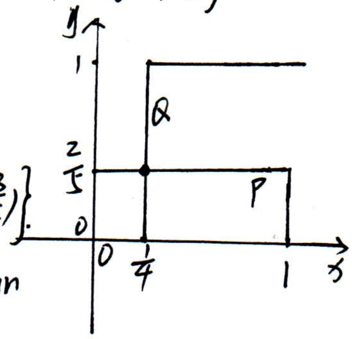
On the other hand,
$$\vec{x}B = (x, 1-x)\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = (2x+2, -2x+3)$$

and
$$\begin{cases} 2x+2 < -2x+3 & \text{if } 0 \le x < \frac{1}{4} \\ 2x+2 = -2x+3 & \text{if } x = \frac{1}{4} \\ 2x+2 > -2x+3 & \text{if } \frac{1}{4} < x \le 1 \end{cases}$$

Hence

$$Q = \left\{ (x,y) : \left(0 \le x < \tfrac{1}{4} \wedge y = 0 \right) \vee \left(x = \tfrac{1}{4} \wedge 0 \le y \le 1 \right) \vee \left(\tfrac{1}{4} < x \le 1 \wedge y = 1 \right) \right\}$$

Now
$$P \cap Q = \left\{ \left(\tfrac{1}{4}, \tfrac{2}{5} \right) \right\}$$



Therefore, the gam has a Nash equilibrium $(\vec{p}, \vec{q}) = \left\{ \left(\frac{1}{4}, \frac{3}{4} \right), \left(\frac{2}{5}, \frac{3}{5} \right) \right\}$

The payoff for row player is $5 - 4 \cdot \frac{2}{5} = \frac{17}{5}$; payoff for column player is $2 \cdot \frac{1}{4} + 2 = \frac{5}{2}$.

(b) Consider $A\vec{y}^{T} = \begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 2+3y \\ 3-2y \end{pmatrix}$

For $2+3y = 3-2y \Rightarrow y = \frac{1}{5}$

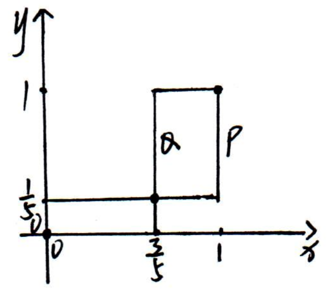
$$P = \left\{ (x,y) : \left(x=0 \wedge 0 \le y < \tfrac{1}{5} \right) \vee \left(0 \le x \le 1 \wedge y = \tfrac{1}{5} \right) \vee \left(x = 1 \wedge \tfrac{1}{5} < y \le 1 \right) \right\}$$

On the other hand, $\vec{x}B = (x, 1-x)\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix} = (1+x, 4-4x)$. For $1+x = 4-4x \Rightarrow x = \frac{3}{5}$.

$$Q = \left\{ (x,y) : \left(0 \le x < \tfrac{3}{5} \wedge y = 0 \right) \vee \left(x = \tfrac{3}{5} \wedge 0 \le y \le 1 \right) \vee \left(\tfrac{3}{5} < x \le 1 \wedge y = 1 \right) \right\}$$

Now $P \cap Q = \left\{ (0,0), \left(\frac{3}{5}, \frac{1}{5} \right), (1,1) \right\}$

Therefore, the game has 3 Nash equilibria:



$\{(0,1),(0,1)\}$, $\{(\frac{3}{5},\frac{2}{5}),(\frac{1}{5},\frac{4}{5})\}$, $\{(1,0),(1,0)\}$

For $(\vec{p},\vec{q})=\{(0,1),(0,1)\}$, the payoff for row player is $\pi(\vec{p},\vec{q})=\vec{p}A\vec{q}^T=(0,1)\begin{pmatrix}5&2\\1&3\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=3$; payoff for column player is $\rho(\vec{p},\vec{q})=\vec{p}B\vec{q}^T=(0,1)\begin{pmatrix}2&0\\1&4\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=4$.

Similarly, the payoff pair of each equilibrium is $(3,4)$, $(\frac{13}{5},\frac{8}{5})$, $(5,2)$ respectively.

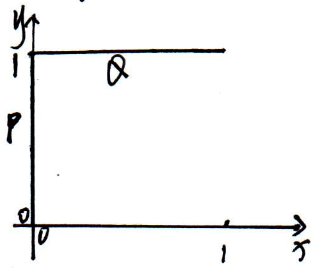
(c) Consider $A\vec{y}^T=\begin{pmatrix}1&2\\5&4\end{pmatrix}\begin{pmatrix}y\\1-y\end{pmatrix}=\begin{pmatrix}2-y\\4+y\end{pmatrix}$

Since $2-y<4+y$ $\forall y\in[0,1]$, $P=\{(0,y):0\le y\le 1\}$

Consider $\vec{x}B=(x,1-x)\begin{pmatrix}5&3\\2&2\end{pmatrix}=(2+3x,2+x)$

Since $2+3x>2+x$ $\forall\, 0<x\le 1$, $Q=\{(x,y):(0<x\le 1\wedge y=1)\vee(x=0\wedge 0\le y\le 1)\}$

Now $P\cap Q=\{(0,y):0\le y\le 1\}$



Therefore, the game has infinite Nash equilibria:

$\{(0,1),(y,1-y)\}$ $\forall y\in[0,1]$.

The corresponding payoff pair of the equilibrium $\{(0,1),(y,1-y)\}$ is $(4+y,2)$, $\forall y\in[0,1]$.

2. (a) Define the map $f: X\to X$

$$(x,y)\longmapsto(-y,x)$$

If f has a fixed point (x_0,y_0), then

$$\begin{cases}-y_0=x_0\\x_0=y_0\end{cases}\Rightarrow\begin{cases}x_0=0\\y_0=0\end{cases}$$

But $(0,0)\notin X$. Hence f has no fixed point.

(b) Define the map $f: X\to X$

$$(x,y,z)\longmapsto(y,-z,x)$$

If $f(x_0,y_0,z_0)=(x_0,y_0,z_0)$, then $x_0=y_0$, $y_0=-z_0$, $z_0=x_0$.

Hence $x_0=y_0=z_0=0$. But $(0,0,0)\notin X$, so f has no fixed point.

(c) Define the map $f: X \to X$

$$(x,y) \mapsto \left(\frac{x+1}{2}, \frac{y}{2}\right)$$

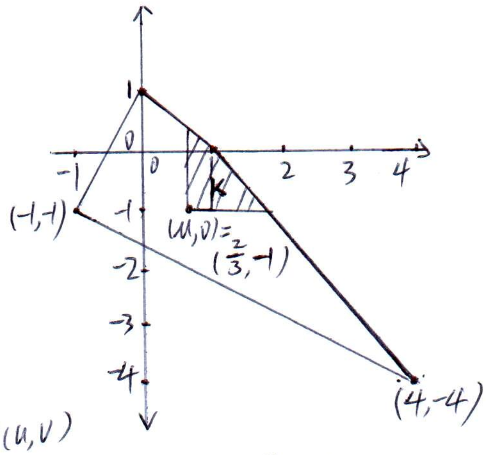
If $f(x_0,y_0)=(x_0,y_0)$, then $\begin{cases} \frac{x_0+1}{2} = x_0 \\ \frac{y_0}{2} = y_0 \end{cases} \Rightarrow \begin{cases} x_0 = 1 \\ y_0 = 0 \end{cases}$

But $(1,0) \notin X$, so f has no fixed point.

3.(a) $A = \begin{pmatrix} 4 & -1 \\ 0 & 1 \end{pmatrix}$, $v_A = \frac{4\times 1 - 0}{4+1-(-1)} = \frac{2}{3}$

$$B^T = \begin{array}{c|c} \begin{pmatrix} -4 & 1 \\ -1 & 0 \end{pmatrix} & \begin{array}{c} \text{min} \\ -4 \\ -1 \end{array} \\ \begin{array}{cc} \text{max} \ -1 & 1 \end{array} & \end{array}, \quad v_{B^T} = -1$$

$(u,v) = \left(\frac{2}{3}, -1\right)$



We need to consider two line segments.

1. The line segment joining $(0,1)$ and $(1,0)$:

The equation is given by $v=-u+1$, the value of $g(u,v)$ along the line segment is $g(u,v) = (u-\frac{2}{3})(-u+1+1) = -u^2+\frac{8}{3}u-\frac{4}{3}$ which attains its maximum at $\left(\frac{4}{3}, -\frac{1}{3}\right)$.

Since the payoff pair lies outside the line segment, it's not the arbitration pair.

2. The line segment joining $(1,0)$ and $(4,-4)$:

The slope of the line joining $(1,0)$ and $(4,-4)$ is $-\frac{4}{3}$. We may solve

$$\begin{cases} v = -\frac{4}{3}(u-1) \\ v+1 = \frac{4}{3}\left(u-\frac{2}{3}\right) \end{cases}$$

which gives $(u,v) = \left(\frac{29}{24}, -\frac{5}{18}\right)$

Since this payoff pair lies on the line segment joining $(1,0)$ and $(4,-4)$, we conclude that the arbitration pair is

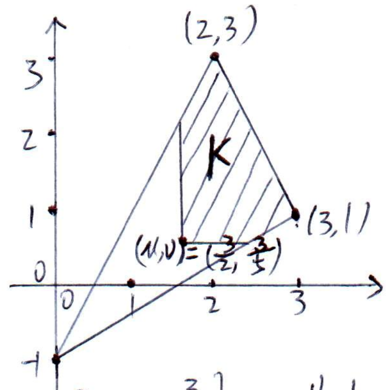
$(\alpha, \beta) = \left(\frac{29}{24}, -\frac{5}{18}\right)$.

(b)

$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$, $v_A = \frac{3\times 2 - 0}{3+2-1} = \frac{3}{2}$

$B^T = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$, $v_{B^T} = \frac{1\times 3 - 0}{1+3-(-1)} = \frac{3}{5}$

$(u,v) = \left(\frac{3}{2}, \frac{3}{5}\right)$



We need to find the payoff pair on $K = \left\{(u,v) \in \mathbb{R} : u \geq \frac{3}{2}, v \geq \frac{3}{5}\right\}$ so that the function $g(u,v) = (u-\frac{3}{2})(v-\frac{3}{5})$ attains its maximum.

The slope of the line joining $(2,3)$ and $(3,1)$ is -2. To find the maximum point of $g(u,v)$ along the line joining $(2,3)$ and $(3,1)$, we may solve

$$\begin{cases} v-1=-2(u-3) \\ v-\frac{3}{5}=2(u-\frac{3}{2}) \end{cases}$$

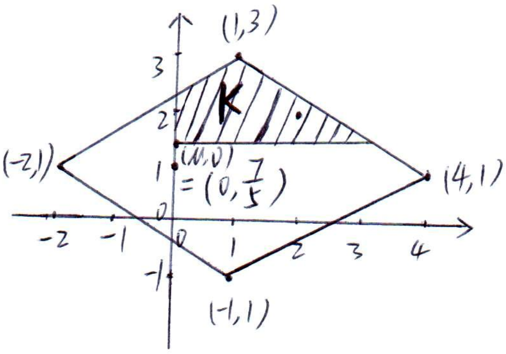
which gives $(u,v)=(\frac{47}{20},\frac{23}{10})$.

Since the payoff pair lies on the line segment joining $(2,3)$ and $(3,1)$, we conclude that the arbitation pair is $(\alpha,\beta)=(\frac{47}{20},\frac{23}{10})$.

(c) $A=\begin{array}{c|c} \begin{pmatrix} 2 & 0 & 1 \\ 4 & -2 & 1 \end{pmatrix} & \begin{array}{c} \text{min} \\ 0 \\ -2 \end{array} \\ \begin{array}{ccc} \text{max } 4 & 0 & 1 \end{array} & \end{array}$, $v_A=0$

$B^T=\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ -1 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$, $v_{B^T}=\frac{2\times 3-(-1)}{2+3-(-1)-1}=\frac{7}{5}$

$(u,v)=(0,\frac{7}{5})$



We need to find the payoff pair on $K=\{(u,v)\in R: u\geq 0, v\geq \frac{7}{5}\}$ so that the function $g(u,v)=u(v-\frac{7}{5})$ attains its maximum.

The slope of the line joining $(1,3)$ and $(4,1)$ is $-\frac{2}{3}$. To find the maximum point of $g(u,v)$ along the line joing $(1,3)$ and $(4,1)$, we may solve

$$\begin{cases} v-1=-\frac{2}{3}(u-4) \\ v-\frac{7}{5}=\frac{2}{3}u \end{cases}$$

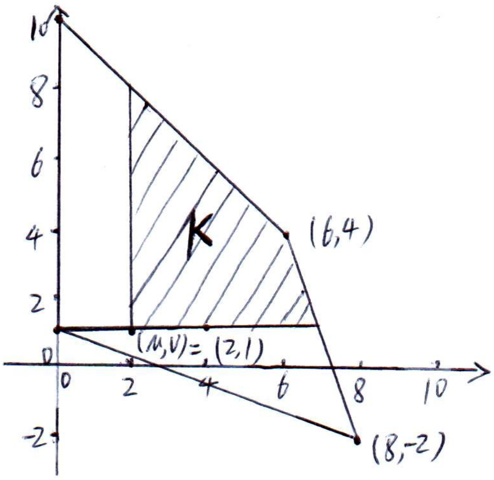
which gives $(u,v)=(\frac{17}{10},\frac{38}{15})$.

Since the payoff pair lies on the line segment joining $(1,3)$ and $(4,1)$, we conclude that the arbitration pair is $(\alpha,\beta)=(\frac{17}{10},\frac{38}{15})$.

(d) $A=\begin{pmatrix} 6 & 0 & 4 \\ 8 & 4 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$, $v_A=\frac{-4\times 4}{-4-4}=2$

$B^T=\begin{array}{c|c} \begin{pmatrix} 4 & -2 \\ 10 & 1 \\ 1 & 1 \end{pmatrix} & \begin{array}{c} \text{min} \\ -2 \\ 1 \\ 1 \end{array} \\ \begin{array}{cc} \text{max } 10 & 1 \end{array} & \end{array}$, $v_{B^T}=1$.

$(u,v)=(2,1)$.



We need to consider two line segments.

1. The line segment joining $(6,4)$ and $(8,-2)$:

   The equation of the line segment is given by $v=-3u+22$. The value of $g(u,v)$ along the line segment is $g(u,v)=(u-2)(v-1)=(u-2)(-3u+22-1)=-3u^2+27u-42$ which attains its maximum at $(\frac{9}{2},\frac{17}{2})$.

   Since this payoff pair lies outside the line segment joining $(6,4)$ and $(8,-2)$ and thus lies outside K, we know that the arbitration pair does not lie on the lie segment joining $(6,4)$ and $(8,-2)$.

2. The line segment joining $(0,10)$ and $(6,4)$:

   The slope of the line joining $(0,10)$ and $(6,4)$ is -1. To find the maximum point of $g(u,v)$ along the line joining $(0,10)$ and $(6,4)$, we may solve

$$\begin{cases} v-10=-u \\ v-1=u-2 \end{cases}$$

   which gives $(u,v)=(\frac{11}{2},\frac{9}{2})$.

   Since this payoff pair lies on the line segment joining $(0,10)$ and $(6,4)$, we conclude that the arbitation pair is

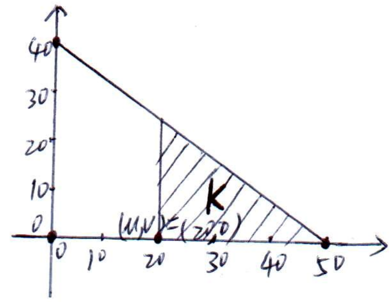
   $(\alpha,\beta)=(\frac{11}{2},\frac{9}{2})$.

4. By the conditions, if we view NTV and CTV as the row player and column player respectively, then the bimatrix game is as follows:

$$(A,B)=\begin{pmatrix} (20,0) & (50,0) \\ (0,40) & (0,0) \end{pmatrix}$$

We use $(u,v)=(v_A,v_{B^T})=(20,0)$ as the status quo point. We need to find the payoff pair on $K=\{(u,v)\in R: u\geq 20, v\geq 0\}$ so that the function $g(u,v)=(u-20)\cdot v$ attains its maximum.

Now any payoff pair (u,v) along the line segment joining $(0,40)$ and $(50,0)$ satisfies



$$v - 40 = -\frac{4}{5}u$$

$$\Rightarrow \quad v = -\frac{4}{5}u + 40$$

Thus $g(u,v) = (u-20)\cdot v = (u-20)(-\frac{4}{5}u+40) = -\frac{4}{5}u^2 + 56u - 800$

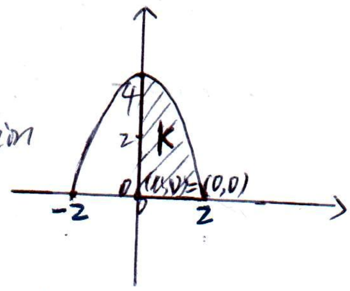
attains its maximum when $u = 35$ and $v = 12$

Since this payoff pair lies on the line segment joining $(0,40)$ and $(50,0)$, the Nash's solution to the bargaining problem with status quo point $(\mu,\nu)=(20,0)$ is

$(\alpha,\beta) = (35, 12)$.

5. (a) We need to find the payoff pair on

$K = \{(u,v) \in R : u \geq 0,\ v \geq 0\}$ so that the function



$g(u,v) = uv$ attains its maximum.

Now any payoff pair (u,v) along the curve $v = 4 - u^2,\ 0 \leq u \leq 2$ satisfies its equation, i.e. $v = 4-u^2,\ 0 \leq u \leq 2$.

Thus $g(u,v) = uv$

$$= -u^3 + 4u \quad , \quad 0 \leq u \leq 2$$

attains its maximum when $u = \frac{2\sqrt{3}}{3}$ and $v = \frac{8}{3}$. (let $g'(u,v) = -3u^2+4 = 0$)

Since this payoff pair lies on the curve $v = 4-u^2,\ 0 \leq u \leq 2$, the arbitration pair of the game with status quo point $(\mu,\nu) = (0,0)$ is

$$(\alpha,\beta) = \left(\frac{2\sqrt{3}}{3}, \frac{8}{3}\right)$$

(b) We need to find the payoff pair on

$K = \{(u,v) \in R : u \geq 0,\ v \geq 1\}$ so that the function $g(u,v) = u(v-1)$ attains its maximum.

Similarly, we get $g(u,v) = u(v-1) = u(4-u^2-1) = -u^3+3u,\ 0 \leq u \leq 2$, which attains its maximum when $u=1$ and $v=3$.

Since this payoff pair lies on the curve $v=4-u^2$, $0\le u\le 2$, the arbitration pair of the game with status quo point $(\mu,\nu)=(0,1)$ is

$$(\alpha,\beta)=(1,3).$$

6. Pf. By the conditions, the function $g(u,v)=(u-\mu)(v-\nu)$ attains its maximum at (α,β) on $K=\{(u,v)\in R: u\geq\mu,\ v\geq\nu\}$, where $\beta=f(\alpha)$.

Hence $g(u,v)=(u-\mu)(f(u)-\nu)$ attains its maximum at $u=\alpha$.

Let $h(u)=g(u,v)=(u-\mu)(f(u)-\nu)$, then

$h'(\alpha)=(f(\alpha)-\nu)+(\alpha-\mu)\cdot f'(\alpha)=0$

$\Rightarrow f'(\alpha)=-\dfrac{f(\alpha)-\nu}{\alpha-\mu}=-\dfrac{\beta-\nu}{\alpha-\mu}$, where $\dfrac{\beta-\nu}{\alpha-\mu}$ is the slope of the line joining (μ,ν) and (α,β).